1.9 Linear Transformations (and matrices)

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"We've seen that natural linear transformations come from matrix multiplication, today, we lean into this fact to explore why, in fact, every linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation."

Def: The $n \times n$ identity matrix is the $n \times n$ matrix whose only nonzero entries are 1s along it's main diagonal.

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$

$e_1\ e_2\ e_3$ in $\mathbb{R}^3$

We label the columns of this matrix $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ in $\mathbb{R}^n$.

Key idea: A linear transformation from $\mathbb{R}^n \to \mathbb{R}^m$ is determined completely by it's action on $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ in $\mathbb{R}^n$.

Ex If $T: \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation that maps $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as follows: $T(\vec{e}_1) = \begin{bmatrix} -1 \\ 2 \\ 3 \\ -5 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 7 \end{bmatrix}$.

Then we can compute the image of any $\vec{x}$ in $\mathbb{R}^2$ (and thus the range of T as well.)

Note: $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x_1 \vec{e}_1 + x_2 \vec{e}_2$; T is linear so

$$T(\vec{x}) = T(x_1\vec{e}_1 + x_2\vec{e}_2) = x_1 T(\vec{e}_1) + x_2 T(\vec{e}_2)$$

$$= \underbrace{x_1 \begin{bmatrix} -1 \\ 2 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 4 \\ 0 \\ 7 \end{bmatrix}}_{\text{Range of } T \text{ is } \operatorname{Span}\{T(\vec{e}_1), T(\vec{e}_2)\}} = \underbrace{\begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 3 & 0 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{T \text{ is a matrix transformation.}} = \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} \vec{x}}_{\text{And the matrix in question is of this form.}}$$

This tells us a lot.

Fact: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there is a unique matrix A s.t. $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$.

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Furthermore, this matrix is of the form

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

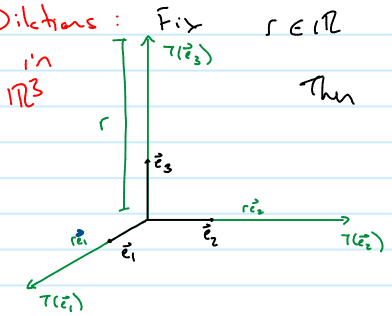
where $\vec{e}_j$ in $\mathbb{R}^n$ has only one nonzero entry: 1 in the j^{th} position.

We call A the standard matrix for the linear transformation T.

Thus, given any description of T, we may find A if and only if we know each image $T(\vec{e}_i)$ for $\vec{e}_1, \dots, \vec{e}_n$ in $\mathbb{R}^n$.

Geometric examples: "Note the examples of the book are posted"

Dilations in $\mathbb{R}^3$: Fix $r \in \mathbb{R}$ and define $T: \mathbb{R}^3 \to \mathbb{R}^3$ to be $T(\vec{x}) = r\vec{x}$.



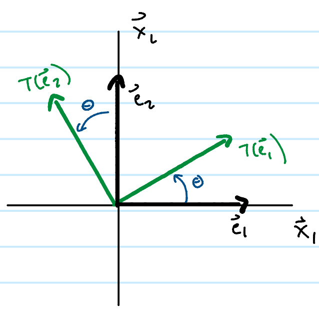
Then $T(\vec{e}_1) = r\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}r\\0\\0\end{bmatrix}$, and $T(\vec{e}_2) = \begin{bmatrix}0\\r\\0\end{bmatrix}$, $T(\vec{e}_3) = \begin{bmatrix}0\\0\\r\end{bmatrix}$.

So the standard matrix of T is

$$A = \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix}.$$

(The image $\vec{x}$ of $\vec{x}$ in $\mathbb{R}^3$ under T is $T(\vec{x}) = A\vec{x}$.)

Rotations of angle θ in $\mathbb{R}^2$:



If $T: \mathbb{R}^2 \to \mathbb{R}^2$ maps $\vec{x}$ to $T(\vec{x})$ by rotating $\vec{x}$ about the origin by the angle θ

then $T(\vec{e}_1) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ and $T(\vec{e}_2) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$

so for any $\vec{x}$, $T(\vec{x}) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\vec{x}$.

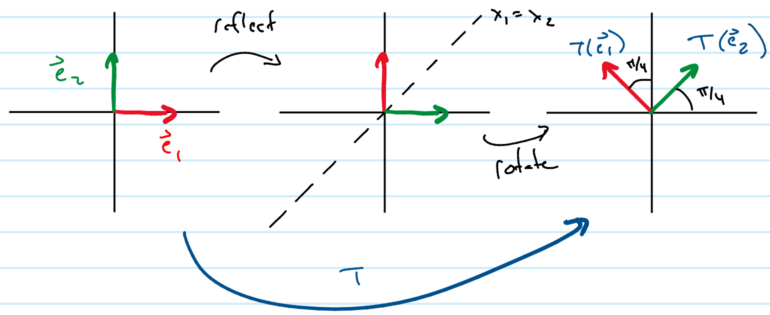
"We consider some more examples from the book.

Do a reflection and show an example

$T(\vec{e}_1) = -\vec{e}_1$, $T(\vec{e}_2) = \vec{e}_2 \Rightarrow A = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$. $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right) = \begin{bmatrix}-3\\4\end{bmatrix} = A \cdot \begin{bmatrix}3\\4\end{bmatrix}$.

Notice we can practice this for all of these examples: a good idea is to hide the standard matrix and find it from the picture, or, cover the picture examples: — good idea is to know the standard matrix and find it from the picture, or, cover the picture and understand what picture the standard matrix implies

Ex) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation which reflects a vector $\vec{x}$ across the line $x_1 = x_2$ before rotating around the origin $\pi/4$ radians.



We see $T(\vec{e}_1) = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$, $T(\vec{e}_2) = \begin{bmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

Thus $T(\vec{x}) = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^2$.

Notice! $\left(\text{rotate by } \pi/4\right) \underset{\text{composition}}{\circ} \left(\text{reflect across } x_1 = x_2\right) = \left(\text{apply } T\right)$

$$\begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \underset{\text{matrix multiplication}}{\cdot} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$$

From the standard matrix of T, we can deduce two important facts about $T: \mathbb{R}^n \to \mathbb{R}^m$

$\vec{x} \xrightarrow{T} \vec{u}$, $\vec{y} \to \vec{v}$; $\mathbb{R}^n$ yes. $\mathbb{R}^m$ $\qquad$ $\vec{x} \to \vec{u}$, $\vec{y} \xrightarrow{T} \vec{u}$; $\mathbb{R}^n$ no. $\mathbb{R}^m$

is T one-to-one?

↳ Distinct $\vec{x}$ and $\vec{y}$ in $\mathbb{R}^n$ have distinct images in $\mathbb{R}^m$.

So, $\underset{\text{in } \mathbb{R}^n}{\vec{x} \neq \vec{y}} \Longrightarrow \underset{\text{in } \mathbb{R}^m}{T(\vec{x}) \neq T(\vec{y})}$

is T onto?

↳ The range of T is all of $\mathbb{R}^m$.

e.g. does T hit only a plane in $\mathbb{R}^3$ or every point in space?

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Fact: If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A, then

1) T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.

2) T is one-to-one if and only if the columns of A are linearly independent.

(Equivalently $T(\vec{x}) = \vec{0}$ is only true for $\vec{0}$.)

↳ So $A\vec{x} = \vec{0}$ has only the trivial solution.

Ex) Let $T: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with standard matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

pivot columns

Is T one-to-one? Notice some column is not a pivot column, so the columns are linearly dependent. So no.

Is T onto? Notice each row has a pivot position. So for each $\vec{b}$ in $\mathbb{R}^3$, $A\vec{x} = \vec{b}$ has a solution. So every $\vec{b}$ in $\mathbb{R}^3$ is the image of some $\vec{x}$ in $\mathbb{R}^4$, i.e.

$$T(\vec{x}) = \vec{b}$$

So yes.

by this, we mean "is the range of T <u>all</u> of the codomain $\mathbb{R}^m$?"